

Preliminary group classification for generalized inviscid Burger's equation

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Abstract

Preliminary group classification for a class of generalized inviscid Burger's equations in the general form $u_t + g(x, u)u_x = f(x, u)$ is given and additional equivalence transformations are found. Adduced results complete and essentially generalize recent works on the subject. A number of new interesting nonlinear invariant models which have non-trivial invariance algebras are obtained. The result of the work is a wide class of equations summarized in table form.

Key words: Burgers' equation, Equivalence transformations, Lie symmetries.

1 Introduction

The theory of Lie groups has greatly influenced diverse branches of mathematics and physics. The main tool of the theory, Sophus Lie's infinitesimal method [5], establishes connection between continuous transformation groups and algebras of their infinitesimal generators. The method leads to many techniques of great significance in studying the group-invariant solutions and conservation laws of differential equations [2, 8, 10]. The approach to the classification of partial differential equations which we propound is, in fact, a synthesis of Lie's infinitesimal method, the use of equivalence transformations and the theory of classification of abstract finite-dimensional Lie algebras.

Historically, Burgers' equation is a fundamental partial differential equation from fluid mechanics [4]. There have been a number of papers that have contributed to the studies of Lie groups of transformations of various classes of Burgers' equation, such as modeling of gas dynamics and traffic flow [11].

The classical form of Burgers' equation is

$$u_t + u u_x = \nu u_{xx}, \quad (1)$$

for viscosity coefficient ν , which describes the evolution of the field $u = u(t, x)$ under nonlinear advection and linear dissipation [3, 10]. When $\nu = 0$, Burgers' equation becomes the inviscid Burgers' equation:

$$u_t + u u_x = 0. \quad (2)$$

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which is a prototype for equations for which the solution can develop discontinuities (shock waves). The generalized non-homogeneous inviscid Burgers equation had introduced in [12] has the form

$$u_t + g(u) u_x = f(u). \quad (3)$$

In this study, we extend equation (3) and consider the following generalized non-homogeneous inviscid Burgers' equation (NIB) in the general form

$$\text{NIB : } u_t + g(x, u) u_x = f(x, u), \quad (4)$$

where $f \neq 0$ and $g \neq 0$ are sufficiently smooth nonconstant functions of variables x, u .

The main results of the present paper are point symmetry and equivalence classification of NIB equations leading to a number of new interesting nonlinear invariant models associated to non-trivial invariance algebras. A complete list of these models are given for a finite-dimensional subalgebra of infinite-dimensional equivalence algebra derived for NIB equations. Reaching to these goals we follow an algorithm explained and performed in references [3, 10]. Our method is similar to the way of [6] for the nonlinear heat conductivity equation $u_t = [E(x, u)u_x]_x + H(x, u)$.

Throughout this paper we assume that each index of a function implies the derivation of the function with respect to it, unless specially stated otherwise.

2 Principal Lie algebra

The symmetry approach to the classification of admissible partial differential equations relies heavily upon the availability of a constructive way of describing transformation groups leaving invariant the form of a given partial differential equation. This is done via the well-known infinitesimal method developed by Sophus Lie [8, 9, 10]. Given a partial differential equation, the problem of investigating its maximal (in some sense) Lie invariance group reduces to solving an over-determined system of linear partial differential equations, called the *determining equations*.

Considering the total space $E = X \times U$ with local coordinate (t, x, u) of independent variables $(t, x) \in X$ and dependent variable $u \in U$, the solution space of Eq. (4), is a subvariety $S_\Delta \subset J^1(\mathbb{R}^2, \mathbb{R})$ of the first order jet bundle of 2-dimensional submanifolds of E . The one-parameter Lie group of infinitesimal transformations on E is as follows

$$\begin{aligned} \tilde{t} &= t + s \xi(t, x, u) + O(s^2), \\ \tilde{x} &= x + s \tau(t, x, u) + O(s^2), \\ \tilde{u} &= u + s \varphi(t, x, u) + O(s^2), \end{aligned} \quad (5)$$

where s is the group parameter and ξ, τ and φ are the (sufficiently smooth) infinitesimals of the transformations for the independent and dependent variables, resp. Thus the corresponding infinitesimal generators have the following general form

$$\mathbf{v} = \xi(t, x, u) \frac{\partial}{\partial t} + \tau(t, x, u) \frac{\partial}{\partial x} + \varphi(t, x, u) \frac{\partial}{\partial u}. \quad (6)$$

According to the invariance condition (see e.g. [1], Theorem 2.36), \mathbf{v} is a point transformation of Eq. (4) if and only if

$$\mathbf{v}^{(1)}[u_t + g u_x - f]\Big|_{Eq. (4)} = 0. \quad (7)$$

In the latter relation, $\mathbf{v}^{(1)}$ is the first order prolongation of vector field \mathbf{v} :

$$\mathbf{v}^{(1)} = \mathbf{v} + \varphi^t(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}) \frac{\partial}{\partial u_t} + \varphi^x(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}) \frac{\partial}{\partial u_x}, \quad (8)$$

in which

$$\varphi^t = \mathbb{D}_t(\mathcal{Q}) + \xi u_{tt} + \tau u_{tx}, \quad (9)$$

$$\varphi^x = \mathbb{D}_x(\mathcal{Q}) + \xi u_{tx} + \tau u_{xx}, \quad (10)$$

and $\mathcal{Q} = \varphi - \xi u_t - \eta u_x$ is the characteristic of vector field \mathbf{v} and the operators

$$\mathbb{D}_J = \partial_J + u_{t,J} \partial u_t + u_{x,J} \partial u_x + \dots$$

(for $J = t, x$) are total derivatives with respect to t and x . Applying $\mathbf{v}^{(1)}$ on Eq. (4) and introducing the relation $u_t = f - g u_x$ to eliminate u_t , we obtain the following determining equation:

$$\begin{aligned} & -\tau g g_x u_x + \tau f g_x - \tau f_x + \varphi g_u u_x - \varphi f_u + \varphi_t + f \varphi_u - g \varphi_u u_x + g \xi_t u_x - f \xi_t \\ & -\tau_t u_x - f^2 \xi_u + f g \xi_u u_x - \tau_u f u_x + g \varphi_x + g \varphi_u u_x - g \tau_x u_x + g^2 \xi_x u_x - f g \xi_x = 0. \end{aligned} \quad (11)$$

In the last relation, since ξ, τ and φ do not depend to variable u_x and its powers, so Eq. (11) is hold if and only if the following equations are fulfilled

$$\begin{aligned} & -\tau g g_x + \varphi g_u + \xi_t g - \tau_t + f g \xi_u - \tau_u f - g \tau_x + g^2 \xi_x = 0, \\ & \varphi_t - \tau f_x - \varphi f_u + \varphi_u f - \xi_t f - \xi_u f^2 + g \varphi_x - f g \xi_x = 0. \end{aligned} \quad (12)$$

But using the fact that $\{1, g, g^2\}$ and $\{1, f, f^2\}$ for nonconstant functions f, g are independent sets (see Lemma 1 of [7]), one can divide Eqs. (12) to the below over-determined system

$$\begin{aligned} & \tau_u = 0, \quad \xi_u = 0, \quad \xi_t - \tau_x - \tau g_x = 0, \quad \varphi g_u - \tau_t = 0, \\ & \xi_x = 0, \quad \varphi_x = 0, \quad \phi_t - \tau f_x - \varphi f_u = 0, \quad \varphi_u - \xi_t = 0. \end{aligned} \quad (13)$$

General solution to this system results in the final form of point infinitesimal generators:

Theorem 1. *Infinitesimal generator of every one-parameter Lie group of point symmetries of NIB equation is*

$$\mathbf{v} = \frac{\partial}{\partial t}. \quad (14)$$

Furthermore, every NIB equation in the form Eq. (4) can not be reduced into an inhomogeneous form of a linear equation.

Table 1: The commutators table for point symmetry algebra of IBE.

	\mathbf{v}_1	\mathbf{v}_2	\mathbf{v}_3	\mathbf{v}_4
\mathbf{v}_1	0	$\mathbf{v}_{F^1 F^2}^1$	0	$-\mathbf{v}_{F_u^1 F^4}^1 + \mathbf{v}_{F_u^1 F^4 g_u}^3$
\mathbf{v}_2	$-\mathbf{v}_{F^1 F^2}^1$	0	$-\mathbf{v}_{F^2 F^3}^3$	$-\mathbf{v}_{F_u^2 F^4}^2$
\mathbf{v}_3	0	$\mathbf{v}_{F^2 F^3}^3$	0	$\mathbf{v}_{F_u^3 F^4}^3$
\mathbf{v}_4	$\mathbf{v}_{F_u^1 F^4}^1 - \mathbf{v}_{F_u^1 F^4 g_u}^3$	$\mathbf{v}_{F_u^2 F^4}^2$	$-\mathbf{v}_{F_u^3 F^4}^3$	0

Proof: The second assertion is a simple result of p. 209 of [8]. \diamond

Let we change the conditions on f, g and permit some of them be zero: Assume that $f = 0$ and $g = g(u)$. In this case, NIB equation reduced to the equation $IBE : u_t + g(u) u_x = 0$ which has recently studied in [7]. Applying the conditions on f, g on determining equation (11) and the same computation as above show that the general form of infinitesimal, represented in Theorem 1 of [7] is not correct. The correct form of infinitesimal generators of IBE is as follows

$$\mathbf{v} = [F^2(u) t + F^1(u)] \frac{\partial}{\partial t} + [F^4(u) g_u t + F^2(u) x + F^3(u)] \frac{\partial}{\partial x} + F^4(u) \frac{\partial}{\partial u}, \quad (15)$$

where F^i s are arbitrary smooth functions of u . One may divides \mathbf{v} to the following vector fields

$$\begin{aligned} \mathbf{v}_1 &:= \mathbf{v}_{F^1}^1 = F^1(u) \frac{\partial}{\partial t}, & \mathbf{v}_2 &:= \mathbf{v}_{F^2}^2 = F^2(u) \left[t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} \right], \\ \mathbf{v}_3 &:= \mathbf{v}_{F^3}^3 = F^3(u) \frac{\partial}{\partial x}, & \mathbf{v}_4 &:= \mathbf{v}_{F^4}^4 = F^4(u) \left[g_u t \frac{\partial}{\partial x} + \frac{\partial}{\partial u} \right]. \end{aligned} \quad (16)$$

The commutative table of these vector fields is given in Table 1.

Also every projective infinitesimal generator introduced in Theorem 2 of [7] has the following correct form

$$\mathbf{v} = [c_1 t + F^1(u)] \frac{\partial}{\partial t} + [c_3 t + c_1 x + c_2] \frac{\partial}{\partial x} + \frac{c_3}{g_u} \frac{\partial}{\partial u}, \quad (17)$$

for $g_u \neq 0$. For $g(u) \equiv \text{const.}$, it has the following modified form

$$\mathbf{v} = [c_1 t + F^1(u)] \frac{\partial}{\partial t} + [c_1 x + c_2] \frac{\partial}{\partial x} + F^2(u) \frac{\partial}{\partial u}. \quad (18)$$

A generalized case of IBE occurs when $f = 0$ and $g = g(x, u)$. Thus we deal with the homogeneous inviscid Burgers' equation. The similar computations show that in this case, the general form of point infinitesimal generators is

$$\mathbf{v} = [F^2(u) t + F^1(u)] \frac{\partial}{\partial t} + e^{-g} \left[F^2 \int e^g dx + F^3(u) \right] \frac{\partial}{\partial x}. \quad (19)$$

3 Equivalence transformations

It is well-known that an equivalence transformation is a non-degenerate change of the variables t, x, u taking any equation of the form (4) into an equation of the same form, generally speaking, with different $f(x, u)$ and $g(x, u)$: The set of all equivalence transformations forms an equivalence group \mathcal{E} : We shall find a continuous subgroup \mathcal{E}_c of it making use of the infinitesimal method [3, 10].

To find the group \mathcal{E}_c we need to determine those infinitesimal generators

$$Y = \xi(t, x, u) \frac{\partial}{\partial t} + \tau(t, x, u) \frac{\partial}{\partial x} + \varphi(t, x, u) \frac{\partial}{\partial u} + \chi(t, x, u, f, g) \frac{\partial}{\partial f} + \eta(t, x, u, f, g) \frac{\partial}{\partial g}, \quad (20)$$

from the invariance conditions of Eq. (4) as the following system

$$\begin{cases} u_t + g(x, u) u_x = f(x, u) \\ f_t = g_t = 0. \end{cases} \quad (21)$$

Here, u and f, g are considered as differential variables: u on the space (t, x) and f, g on the extended space (t, x, u) . The coordinates ξ, τ, φ are sought as functions of t, x, u while the coordinates χ, η are sought as functions of t, x, u, f, g . The invariance conditions of the system (21) are

$$\begin{cases} \tilde{Y}[u_t + g(x, u) u_x - f(x, u)] = 0, \\ \tilde{Y}[f_t] = \tilde{Y}[g_t] = 0, \end{cases} \quad (22)$$

where \tilde{Y} is the prolongation of (20) to the first order jet space of differential variables t, x, u, f, g which one may represent as follows

$$\tilde{Y} = Y + \varphi^t \frac{\partial}{\partial u_t} + \varphi^x \frac{\partial}{\partial u_x} + \chi^t \frac{\partial}{\partial f_t} + \eta^t \frac{\partial}{\partial g_t}. \quad (23)$$

In this relation φ^t and φ^x are the same with those introduced in section 2 and

$$\chi^t = \tilde{\mathcal{D}}_t(\chi) - f_t \tilde{\mathcal{D}}_t(\xi) - f_x \tilde{\mathcal{D}}_t(\tau) - f_u \tilde{\mathcal{D}}_t(\varphi) = \tilde{\mathcal{D}}_t(\chi) - f_x \tilde{\mathcal{D}}_t(\tau) - f_u \tilde{\mathcal{D}}_t(\varphi), \quad (24)$$

$$\eta^t = \tilde{\mathcal{D}}_t(\eta) - g_t \tilde{\mathcal{D}}_t(\xi) - g_x \tilde{\mathcal{D}}_t(\tau) - g_u \tilde{\mathcal{D}}_t(\varphi) = \tilde{\mathcal{D}}_t(\eta) - g_x \tilde{\mathcal{D}}_t(\tau) - g_u \tilde{\mathcal{D}}_t(\varphi), \quad (25)$$

where we consider

$$\tilde{\mathcal{D}}_t := \frac{\partial}{\partial t} + f_t \frac{\partial}{\partial f} + g_t \frac{\partial}{\partial g} = \frac{\partial}{\partial t}. \quad (26)$$

Substituting relations (24)-(26) in (23) and then applying it on the last two equations of (22) we find that

$$\chi_t - f_x \tau_t - f_u \varphi_t = 0, \quad (27)$$

$$\eta_t - g_x \tau_t - g_u \varphi_t = 0. \quad (28)$$

Since the latter equations are hold for every f and g , so we lead to the following facts

$$\tau_t = 0, \quad \varphi_t = 0, \quad \chi_t = 0, \quad \eta_t = 0. \quad (29)$$

Table 2: The commutators table for equivalence symmetry algebra of NIB.

	Y_1	Y_2	Y_3
Y_1	0	$-Y_{\tau \xi_x}^1$	$-Y_{\varphi \xi_u}^1$
Y_2	$Y_{\tau \xi_x}^1$	0	$Y_{\tau \varphi_x}^3 - Y_{\varphi \tau_u}^2$
Y_3	$Y_{\varphi \xi_u}^1$	$Y_{\varphi \tau_u}^2 - Y_{\tau \varphi_x}^3$	0

By effecting (23) on the first equation of (22), we have

$$-\chi + \eta u_x + \varphi^t + g \varphi^x = 0. \quad (30)$$

Substituting φ^t, φ^x , introducing $u_t = -g u_x + f$ to eliminate u_t and using the fact that in the derived relation u_x and its powers are free variables, finally we tend to the below system

$$f g \xi_u + g \xi_t - f \tau_u - g \tau_x + g^2 \xi_x + \eta = 0, \quad (31)$$

$$-\chi + f \varphi_u - f \xi_t - f^2 \xi_u + g \varphi_x - f g \xi_x = 0. \quad (32)$$

The general solution to Eqs. (23), (23) and (23) is

$$\xi = \xi(t, x, u), \quad \eta = \tau(x, u), \quad \varphi = \varphi(x, u) \quad (33)$$

$$\chi = g \varphi_x - f g \xi_x - f \xi_t + f \varphi_u - f^2 \xi_u, \quad (34)$$

$$\eta = f \tau_u + g \tau_x - g \xi_t - g^2 \xi_x - f g \xi_u. \quad (35)$$

After utilizing these relations in Y , one may divide Y to the following vector fields

$$\begin{aligned} Y_1 &:= Y_\xi^1 = \xi \frac{\partial}{\partial t} - f[g \xi_x + \xi_t + f \xi_u] \frac{\partial}{\partial f} - g[f \xi_u + \xi_t + g \xi_x] \frac{\partial}{\partial g}, \\ Y_2 &:= Y_\tau^2 = \tau \frac{\partial}{\partial x} + [g \tau_x + f \tau_u] \frac{\partial}{\partial g}, \\ Y_3 &:= Y_\varphi^3 = \varphi \frac{\partial}{\partial u} + [g \varphi_x + f \varphi_u] \frac{\partial}{\partial f}. \end{aligned} \quad (36)$$

The commutators table of Y_i s is given in Table 2.

4 Preliminary group classification

In diverse applications of symmetry analysis, the equivalence symmetry is handled to extend the principal Lie algebra to the equivalence algebra $\mathcal{L}_{\mathcal{F}}$ when some further equations under considerations are taken. These extensions are called \mathcal{F} -extensions of the principal Lie algebra. The classification of all non-equivalent equations (with respect to a given equivalence group $G_{\mathcal{F}}$) admitting \mathcal{F} -extensions of the principal Lie algebra is called a *preliminary group classification*.

In the general, $G_{\mathcal{F}}$ is not necessarily the largest equivalence group. In the following, we consider a subgroup of the group of all equivalence transformations that has a finite-dimensional subalgebra (desirably as large as possible) of an infinite-dimensional algebra

Table 3: Commutators table for \mathcal{L}_{10}

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}
X_1	0	0	0	X_1	0	0	0	0	0	0
X_2	0	0	0	0	X_1	0	X_2	0	X_3	0
X_3	0	0	0	0	0	X_1	0	X_2	0	X_3
X_4	$-X_1$	0	0	0	$-X_4$	$-X_5$	0	0	0	0
X_5	0	$-X_1$	0	X_4	0	0	$-X_4$	$-X_5$	0	0
X_6	0	0	$-X_1$	X_5	0	0	0	0	$-X_4$	$-X_5$
X_7	0	$-X_2$	0	0	X_4	0	0	$-X_7$	X_9	0
X_8	0	0	$-X_2$	0	X_5	0	X_7	0	$X_{10} - X_6$	X_7
X_9	0	$-X_3$	0	0	0	X_4	$-X_9$	$X_6 - X_{10}$	0	X_9
X_{10}	0	0	$-X_3$	0	0	X_5	0	$-X_7$	$-X_9$	0

with basis (36) and use it for a preliminary group classification. We select the subalgebra \mathcal{L}_{10} , generated by the following vector fields

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial t}, & X_2 &= \frac{\partial}{\partial x}, & X_3 &= \frac{\partial}{\partial u}, \\
X_4 &= t \frac{\partial}{\partial t} - f \frac{\partial}{\partial f} - g \frac{\partial}{\partial g}, & X_5 &= x \frac{\partial}{\partial t} - f g \frac{\partial}{\partial f} - g^2 \frac{\partial}{\partial g}, \\
X_6 &= u \frac{\partial}{\partial t} - f^2 \frac{\partial}{\partial f} - f g \frac{\partial}{\partial g}, & X_7 &= x \frac{\partial}{\partial x} + g \frac{\partial}{\partial g}, \\
X_8 &= u \frac{\partial}{\partial x} + f \frac{\partial}{\partial g}, & X_9 &= x \frac{\partial}{\partial u} + g \frac{\partial}{\partial f}, \\
X_{10} &= u \frac{\partial}{\partial u} + f \frac{\partial}{\partial f},
\end{aligned} \tag{37}$$

The commutator and adjoint representations of \mathcal{L}_{10} are listed in Tables 3 and 4.

It is well known that the problem of the construction of the optimal system of solutions is equivalent to that of the construction of the optimal system of subalgebras [8, 9, 10]. Here, we determine a list (an optimal system) of conjugacy inequivalent subalgebras with the property that any other subalgebra is equivalent to a unique member of the list under some element of the adjoint representation i.e. $\bar{\mathfrak{h}} \text{Ad}(g) \mathfrak{h}$ for some g of a considered Lie group. The adjoint action is given by the Lie series

$$\text{Ad}(\exp(s Y_i)) Y_j = Y_j - s [Y_i, Y_j] + \frac{s^2}{2} [Y_i, [Y_i, Y_j]] - \dots, \tag{38}$$

Then we will deal with the construction of the optimal system of subalgebras of \mathcal{L}_{10} .

Table 4: Adjoint table for \mathcal{L}_{10}

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}
X_1	X_1	X_2	X_3	$X_4 - sX_1$	X_5	X_6	X_7	X_8	X_9	X_{10}
X_2	X_1	X_2	X_3	X_4	$X_5 - sX_1$	X_6	$X_7 - sX_2$	X_8	$X_9 - sX_3$	X_{10}
X_3	X_1	X_2	X_3	X_4	X_5	$X_6 - sX_1$	X_7	X_8	X_9	$X_{10} - sX_3$
X_4	$e^s X_1$	X_2	X_3	X_4	$X_5 + sX_4$	$X_6 + sX_5 - \frac{s^2}{2} X_4$	X_7	$X_8 - sX_2$	X_9	X_{10}
X_5	X_1	$X_2 + sX_1$	X_3	$e^{-s} X_4$	X_5	X_6	$X_7 + (e^s - 1)X_4$	$X_8 + sX_5$	X_9	X_{10}
X_6	X_1	X_2	$X_3 + sX_1$	$X_4 - sX_5$	X_5	X_6	X_7	X_8	$X_9 + sX_4 - \frac{s^2}{2} X_5$	$X_{10} + sX_5$
X_7	X_1	$e^s X_2$	X_3	X_4	$X_5 - sX_4$	X_6	X_7	$X_8 + sX_7$	$e^{-s} X_9$	X_{10}
X_8	X_1	X_2	$X_3 + sX_2$	X_4	$e^{-s} X_5$	X_6	$e^{-s} X_7$	X_8	$X_9 - s(X_{10} - X_6) + (e^{-s} + s - 1)X_7$	$X_{10} + (e^{-s} - 1)X_7$
X_9	X_1	$X_2 + sX_3$	X_3	X_4	X_5	$X_6 - sX_4$	$X_7 + sX_9$	$X_8 + s(X_6 - X_{10}) + \frac{s^2}{2}(X_9 - X_4)$	X_9	$X_{10} - sX_9$
X_{10}	X_1	X_2	$e^s X_3$	X_4	X_5	$X_6 - sX_5$	X_7	$X_8 + sX_7$	$e^s X_9$	X_{10}

Theorem 2. An optimal system of one-dimensional Lie subalgebras NIB equation in the form (4) is provided by those generated by

$$\begin{aligned}
1) \quad A^1 &= X_1, & 11) \quad A^{11} &= \gamma_1 X_6 + X_{10} \\
2) \quad A^2 &= X_2, & 12) \quad A^{12} &= \eta_1 X_8 + X_{10}, \\
3) \quad A^3 &= X_3, & 13) \quad A^{13} &= X_1 + \eta_2 X_8 + X_{10}, \\
4) \quad A^4 &= X_4, & 14) \quad A^{14} &= X_2 + \beta_1 X_4 + X_{10}, \\
5) \quad A^5 &= X_5, & 15) \quad A^{15} &= X_2 + \gamma_2 X_6 + X_{10}, \\
6) \quad A^6 &= X_6, & 16) \quad A^{16} &= \alpha_1 X_3 + \gamma_3 X_6 + X_7, \\
7) \quad A^7 &= X_3 + X_4, & 17) \quad A^{17} &= \alpha_2 X_3 + \gamma_4 X_6 + X_8, \\
8) \quad A^8 &= X_3 + X_5, & 18) \quad A^{18} &= \gamma_5 X_6 + \eta_3 X_8 + X_9, \\
9) \quad A^9 &= X_3 + X_6, & 19) \quad A^{19} &= X_4 + \eta_4 X_8 + X_{10}, \\
10) \quad A^{10} &= X_4 + X_{10}, & 20) \quad A^{20} &= \gamma_6 X_6 + \eta_5 X_8 + X_{10},
\end{aligned} \tag{39}$$

where $\alpha_i, \beta_1, \gamma_k, \eta_l$ ($1 \leq i \leq 2, 1 \leq k \leq 6, 1 \leq l \leq 5$) are arbitrary constants.

Proof: Consider the symmetry algebra \mathcal{L}_{10} of Eq. (4), whose adjoint representation was determined in Table 4. Given a nonzero vector

$$X = \sum_{i=1}^{10} a_i X_i, \tag{40}$$

we simplify X by eliminating as many of the coefficients a_i as possible by use of judicious applications of adjoint maps to X . We perform the process through following cases:

Case a. Let $a_{10} \neq 0$. Scaling X if necessary, we can assume that $a_{10} = 1$:

$$X = \sum_{i=1}^9 a_i X_i + X_{10}. \tag{41}$$

When $a_3 = 0$ or when $a_3 \neq 0$ by applying $\text{Ad}(\exp(a_3 X_3))$ on X to cancel the coefficient of X_3 , one can reduce X to

$$X' = a_1 X_1 + a_2 X_2 + a_4 X_4 + \cdots + X_{10}. \tag{42}$$

Now if $a_5 = 0$ or if $a_5 \neq 0$ by effecting $\text{Ad}(\exp(-a_5 X_5))$ on X' , we can make the coefficient of X_5 vanish:

$$X'' = a_1 X_1 + a_2 X_2 + a_4 X_4 + a_6 X_6 + \cdots + X_{10}. \tag{43}$$

We can also cancel the coefficient of X_7 . In the case which it is nonzero, for $a_7 > 1$ by applying $\text{Ad}(\exp(\ln(a_7 - 1) X_7))$, for $a_7 = 1$ by applying $\text{Ad}(\exp(-\ln(2) X_7))$ and for $a_7 < 1$ by applying $\text{Ad}(\exp(-\ln(1 - a_7) X_7))$ on X'' we change this coefficient to zero:

$$X''' = a_1 X_1 + a_2 X_2 + a_4 X_4 + a_6 X_6 + a_8 X_8 + a_9 X_9 + X_{10}. \quad (44)$$

Furthermore, for different values of a_9 , when it is either zero or nonzero, this coefficient can be vanished; when $a_9 \neq 0$ we act $\text{Ad}(\exp(a_9 X_9))$ on X''' to eliminate a_9 .

Case a1. Let $a_8 \neq 0$ the additional action of $\text{Ad}(\exp(\frac{a_2}{a_8} X_4))$ change X to the following form which the coefficient of a_2 is deleted

$$X'''' = a_1 X_1 + a_4 X_4 + a_6 X_6 + a_8 X_8 + a_9 X_9 + X_{10}. \quad (45)$$

Case a1-1. For $a_6 \neq 0$ we effect $\text{Ad}(\exp(\frac{a_1}{a_6} X_3))$. Then the action of $\text{Ad}(\exp(\frac{a_4}{a_6} X_9))$ for $a_4 \neq 0$ or when $a_4 = 0$ we lead to the following form

$$a_6 X_6 + a_8 X_8 + X_{10}. \quad (46)$$

Now, further simplification is impossible and every one-dimensional subalgebra generated by an X with $a_{10}, a_8, a_6 \neq 0$ is equivalent to the subalgebra spanned by (47). this vector field was introduced in part 20 of the theorem.

Case a1-2. If we change the condition of Case a1-1 to $a_6 = 0$, the action of $\text{Ad}(\exp(\frac{a_1}{a_4} X_1))$ for $a_4 \neq 0$ and then the action of $\text{Ad}(\exp(-\ln(\frac{1}{a_4}) X_5))$ change X'''' to the below final form (part 19 of the theorem)

$$a_4 X_4 + a_8 X_8 + X_{10}. \quad (47)$$

In this case if $a_4 = 0$ we can effect $\text{Ad}(\exp(\ln(\frac{1}{a_1}) X_4))$ for $a_1 \neq 0$ to reach to part 13 of the theorem, while $a_1 = 0$ suggests part 12.

Case a2. Let a_8 in Case a1 be zero.

Case a2-1. Moreover, if $a_6 \neq 0$ by effecting $\text{Ad}(\exp(\frac{a_1}{a_6} X_3))$ and then $\text{Ad}(\exp(\frac{a_4}{a_6} X_9))$ we tend to the form

$$a_2 X_2 + a_6 X_6 + X_{10}. \quad (48)$$

In this form when $a_2 \neq 0$ the adjoint action of $\text{Ad}(\exp(\ln(\frac{1}{a_2}) X_7))$ and when $a_2 = 0$ we find sections 15 and 11 of the theorem resp.

Case a2-1. Let $a_6 = 0$. Then the coefficient a_1 for either $a_1 \neq 0$ by applying $\text{Ad}(\exp(\ln(\frac{1}{a_2}) X_7))$ or $a_1 = 0$ will be zero:

$$\tilde{X} = a_2 X_2 + a_4 X_4 + X_{10}. \quad (49)$$

Furthermore, the application of $\text{Ad}(\exp(\ln(\frac{1}{a_2}) X_7))$ for $a_2 \neq 0$ introduces the vector field $X_2 + a_4 X_4 + x_{10}$, while for $a_2 = 0$ by effecting $\text{Ad}(\exp(-\ln(\frac{1}{a_4}) X_5))$ to change the coefficient of X_4 equal to 1, we lead to the case $X_4 + X_{10}$.

The remaining one-dimensional subalgebras are spanned by vectors of Case a, where $a_{10} = 0$. Let $F_i^s(X) = \text{Ad}(\exp(s X_i)X)$ be a linear map, for $i = 1, \dots, 10$ and every $X \in \mathcal{L}_{10}$. We continue the classification by a similar method as above.

Case b. Let $a_9 \neq 0$, we scale to make $a_9 = 1$. One can set the coefficients of $X_3, X_2, X_5, X_7, X_1, X_4$ as zero (when each of the coefficient is zero we eliminate it from the list) by effecting $F_2^{a_3}, F_4^{a_2}, F_5^{-a_5}, F_7^{-a_7}, F_3^{a_1}, F_9^{a_4}$ resp. Thus we tend to section 18.

Case c. For $a_{10} = a_9 = 0, a_8 \neq 0$ and assuming $a_8 = 1$, we can make the coefficients of X_2, X_5, X_7, X_1, X_4 vanish (when those coefficients are not zero) by applying $F_4^{a_2}, F_5^{-a_5}, F_5^{-a_5}, F_7^{-a_7}, F_1^{a_1}, F_9^{a_4}$ resp. This results in section 17.

Case d. Let $a_{10} = a_9 = a_8 = 0, a_7 \neq 0$ and scale if is necessary to have $a_7 = 1$, we can cancel the coefficients of X_2, X_1, X_4, X_5 by actions of $F_2^{a_2}, F_3^{a_1}, F_9^{a_4}, F_{10}^{a_5}$ resp. to have section 16.

Case e. Let $a_{10} = \dots = a_7 = 0, a_6 \neq 0$. We can suppose that $a_6 = 1$. Applying $F_3^{a_1}, F_9^{a_4}, F_{10}^{a_5}, F_8^{-a_2}$ on X we can changed the coefficients of X_1, X_4, X_5, X_2 to zero (if they are not zero). Then if $a_3 \neq 0$ by acting $\text{Ad}(\exp(\ln(\frac{1}{a_3}) X_{10}))$ we find section 9 and when $a_3 = 0$ we lead to section 6.

Case f. Suppose that $a_{10} = \dots = a_6 = 0, a_5 \neq 0$. By assuming $a_5 = 1$, we can make the coefficients of X_1, X_4, X_2 vanish by $F_2^{a_1}, F_4^{-a_4}, F_8^{-a_2}$. Then if $a_3 \neq 0$ we apply $\text{Ad}(\exp(\ln(\frac{1}{a_3}) X_6))$ to give section 8 and for $a_3 = 0$ we have section 5.

Case g. Consider the case which $a_{10} = \dots = a_5 = 0, a_4 \neq 0$. We can assume that $a_4 = 1$. we can make the coefficients of X_1, X_2 vanish by $F_1^{a_1}, F_8^{-a_2}$. In addition for $a_3 \neq 0$, scaling the coefficient of X_3 by $\text{Ad}(\exp(\ln(\frac{1}{a_3}) X_{10}))$ we have section 7, while $a_3 = 0$ leads to section 4.

Case h. For $a_{10} = \dots = a_4 = 0, a_3 \neq 0$ and assuming $a_3 = 1$, the coefficients of X_1, X_2 will be vanished by the actions of $F_6^{-a_1}, F_8^{-a_2}$ and X reduces to section 3.

Case i. If $a_{10} = \dots = a_3 = 0, a_2 \neq 0$. Scaling $a_2 = 1$, by effecting $F_5^{-a_1}$ the coefficient of X_1 will be canceled which suggests section 2.

Case j. Finally when $a_{10} = \dots = a_2 = 0$ we have section 1 of the theorem.

There is not any more possible case for studying and the proof is complete. \diamond

Now, since f, g in Eq. (4) are functions of variables x, u , so we project their derived optimal system to the space (x, u, f, g) . The nonzero vector fields of (x, u) -projections

of (39) are as follows

$$\begin{aligned}
1) \quad & Z^1 = A^2 = \partial_x, \\
2) \quad & Z^2 = A^3 = u \partial_u, \\
3) \quad & Z^3 = A^4 = f \partial_f + g \partial_g, \\
4) \quad & Z^4 = A^5 = g[f \partial_f + g \partial_g], \\
5) \quad & Z^5 = A^6 = f[f \partial_f + g \partial_g], \\
6) \quad & Z^6 = A^7 = \partial_u - f \partial_f - g \partial_g, \\
7) \quad & Z^7 = A^8 = \partial_u - g[f \partial_f + g \partial_g], \\
8) \quad & Z^8 = A^9 = \partial_u - f[f \partial_f + g \partial_g], \\
9) \quad & Z^9 = A^{10} = u \partial_u - g \partial_g, \\
10) \quad & Z^{10} = A^{11} = u \partial_u + (1 - \gamma_1 f) f \partial_f - f g \partial_g, \\
11) \quad & Z^{11} = A^{12} = A^{13} = \eta_1 u \partial_x + u \partial_u + f \partial_f + \eta_1 g \partial_g, \\
12) \quad & Z^{12} = A^{14} = \partial_x + u \partial_u + (1 - \beta_1) f \partial_f - \beta_1 g \partial_g, \\
13) \quad & Z^{13} = A^{15} = \partial_x + u \partial_u + f[(1 - \gamma_2 f) \partial_f - \gamma_2 g \partial_g], \\
14) \quad & Z^{14} = A^{16} = x \partial_x - \gamma_3 f^2 \partial_f + (1 - \gamma_3 f) g \partial_g, \\
15) \quad & Z^{15} = A^{17} = u \partial_x + \alpha_2 \partial_u - \gamma_4 f^2 \partial_f + (1 - \gamma_4 g) f \partial_g, \\
16) \quad & Z^{16} = A^{18} = \eta_3 u \partial_x + (x + \gamma_5 u) \partial_u + (g - \gamma_5 f^2) \partial_g, \\
17) \quad & Z^{17} = A^{19} = \eta_4 u \partial_x + u \partial_u + (\eta_4 f - g) \partial_g, \\
18) \quad & Z^{18} = A^{20} = \eta_5 u \partial_x + u \partial_u + f[(1 - \gamma_6 f) \partial_f + (\eta_5 - \gamma_6 g) \partial_g].
\end{aligned} \tag{50}$$

According to paper 7 of [3] we conclude that

Proposition 3. *Let $\mathcal{L}_m := \langle X_i : i = 1, \dots, m \rangle$ be an m -dimensional algebra. Denote by A^i ($i = 1, \dots, s$, $0 < s \leq m$, $s \in \mathbf{N}$) an optimal system of one-dimensional subalgebras of \mathcal{L}_m and by Z^i ($i = 1, \dots, t$, $0 < t \leq s$, $t \in \mathbf{N}$) the projections of A^i , i.e., $Z^i = \text{pr}(A^i)$. If equations*

$$g = g(x, u), \quad f = f(x, u), \tag{51}$$

are invariant with respect to the optimal system Z^i then the equation

$$u_t + g(x, u) u_x = f(x, u), \tag{52}$$

admits the operators $Y^i = \text{projection of } A^i \text{ on } (t, x, u)$.

Proposition 4. *Let Eq. (52) and the equation*

$$u_t + \overline{g}(x, u) u_x = \overline{f}(x, u), \tag{53}$$

be constructed according to Proposition 3 via optimal systems Z^i and \overline{Z}^i resp. If the subalgebras spanned on the optimal systems Z^i and \overline{Z}^i resp. are similar in \mathcal{L}_m , then Eqs. (52) and (53) are equivalent with respect to the equivalence group G_m generated by \mathcal{L}_m .

Regarding to Propositions 3 and 4, now our task is the investigation for all non-equivalent equations in the form of Eq. (4) which admit \mathcal{E} -extensions of the principal Lie algebra $\mathcal{L}_{\mathcal{E}}$, by one dimension, that are equations of the form (4) such that they admit,

together with the one basic operator (14) of \mathcal{L}_1 , also a second operator $X^{(2)}$. In each case which this extension occurs, we indicate the corresponding coefficients f, g and the additional operator $X^{(2)}$.

By using the algorithm for operators Z^i ($i = 1, \dots, 18$) to f, g one can find vector fields $X^{(2)}$ s. Let we consider the following examples.

We select the operator

$$Z^5 = f^2 \frac{\partial}{\partial f} + f g \frac{\partial}{\partial g}. \quad (54)$$

The characteristic equations are

$$\frac{df}{f^2} = \frac{dg}{f g}, \quad (55)$$

so the invariants of Z^5 are

$$I_1 = x, \quad I_2 = u, \quad I_3 = \frac{f}{g}. \quad (56)$$

In this case there are no invariant equations because the necessary condition for existence of invariant solutions is not fulfilled (see [10], Section 19.3), that is, invariants (56) cannot be solved with respect to f and g since I_2 is not an invariant function of I_1 . Similar results are hold for vector fields Z^3 and Z^4 .

Let perform the algorithm for another example, by considering vector filed

$$Z^{14} = x \frac{\partial}{\partial x} - \gamma_3 f^2 \frac{\partial}{\partial f} + (1 - \gamma_3 f) g \frac{\partial}{\partial g}, \quad (57)$$

then the characteristic equations corresponding to Z^{14} is

$$\frac{dx}{x} = \frac{df}{-\gamma_3 f^2} = \frac{dg}{(1 - \gamma_3 f)g}, \quad (58)$$

which determines invariants for $\gamma_3 \neq 0$. Invariants can be taken in the following form

$$I_1 = u, \quad I_2 = f - \frac{1}{\gamma_3 \ln x}, \quad I_3 = \frac{g}{x^{1-\gamma_3 f}}. \quad (59)$$

From the invariance equations we can write

$$I_2 = \Phi(I_1), \quad I_3 = \Psi(I_1). \quad (60)$$

It results in the forms

$$f = \Phi(\lambda) + \frac{1}{\gamma_3 \ln x}, \quad g = x^{1-\gamma_3[\Phi(\lambda) + \frac{1}{\gamma_3 \ln x}]} \Psi(\lambda), \quad (61)$$

where $\lambda = u$. For the case which $\gamma_3 = 0$, Z_{14} is in the form $x \partial_x + g \partial_g$ and similar way shows that $f = \Phi(\lambda)$ and $g = x \Psi(\lambda)$ when $\lambda = u$.

Table 5: The result of the classification

N	Z	Invariant λ	Equation	Additional operator $X^{(2)}$
1	Z^1	u	$u_t + \Psi u_x = \Phi$	$\frac{\partial}{\partial x}$
2	Z^2	x	$u_t + \Psi u_x = \Phi$	$\frac{\partial}{\partial u}$
3	Z^6	x	$u_t + e^{\Psi-u} u_x = e^{\Phi-u}$	$t \frac{\partial}{\partial t} + \frac{\partial}{\partial u}$
4	Z^7	x	$u_t + [\Psi + 1/u] u_x = [\Psi + 1/u] \Phi$	$x \frac{\partial}{\partial t} + \frac{\partial}{\partial u}$
5	Z^8	x	$u_t + [\Phi + 1/u] \Psi u_x = \Phi + 1/u$	$u \frac{\partial}{\partial t} + \frac{\partial}{\partial u}$
6	Z^9	x	$u_t + u \Psi u_x = \Phi$	$t \frac{\partial}{\partial t} + u \frac{\partial}{\partial u}$
7	Z^{10}	x	$u_t \pm (\gamma_1 f - 1) \Psi u_x = \frac{u \Phi}{1 + \gamma_1 u \Phi}$	$u [\gamma_1 \frac{\partial}{\partial t} + \frac{\partial}{\partial u}]$
8	Z^{11}	$x - \eta_1 u$	$u_t + [\Psi - \eta_1 u \Phi] u_x = u \Phi$	$u [\eta_1 \frac{\partial}{\partial x} + \frac{\partial}{\partial u}],$ $\frac{\partial}{\partial t} + u [\eta_1 \frac{\partial}{\partial x} + \frac{\partial}{\partial u}]$
9	Z^{12}	$\ln u - x$	$u_t + u^{\beta_1} \Psi u_x = u^{1-\beta_1} \Phi$	$\beta_1 t \frac{\partial}{\partial t} + \frac{\partial}{\partial x} + u \frac{\partial}{\partial u}$
10	$Z^{13}(\gamma_2 \neq 0)$	$\ln u - x$	$u_t + [(\frac{\gamma_2 u \Phi}{\gamma_2 u \Phi - 1} - 1) \Psi]^{\frac{1}{\gamma_2}} u_x = \frac{\gamma_2 u \Phi}{\gamma_2 u \Phi - 1}$	$\gamma_2 u \frac{\partial}{\partial t} + \frac{\partial}{\partial x} + u \frac{\partial}{\partial u}$
11	$Z^{13}(\gamma_2 = 0)$	$\ln u - x$	$u_t + \Psi u_x = u \Phi$	$\frac{\partial}{\partial x}$
12	$Z^{14}(\gamma_3 \neq 0)$	u	$u_t + x^{1-\gamma_3[\Phi + \frac{1}{\gamma_3 \ln x}]} \Psi u_x = \Phi + \frac{1}{\gamma_3 \ln x}$	$\gamma_3 u \frac{\partial}{\partial t} + x \frac{\partial}{\partial x}$
13	$Z^{14}(\gamma = 0)$	u	$u_t + x \Psi u_x = \Phi$	$x \frac{\partial}{\partial x}$
14	$Z^{15}(\alpha_2 = \gamma_4 = 0)$	x	$u_t + [\Psi - \frac{x}{u}] \Phi u_x = \Phi$	$u \frac{\partial}{\partial x}$
15	$Z^{15}(\alpha_2 \neq 0, \gamma_4 = 0)$	$\frac{u}{\alpha_2} - x$	$u_t + [\Psi - \frac{x}{u}] \Phi u_x = \Phi$	$u \frac{\partial}{\partial x} + \alpha_2 \frac{\partial}{\partial u}$
16	$Z^{15}(\alpha_2 = 0, \gamma_4 \neq 0)$	u	$u_t + [\frac{1}{\gamma_4} \pm (\frac{1}{\gamma_4} \Phi - \frac{u}{x}) \Psi] u_x = \Phi - \frac{\gamma_4 u}{x}$	$\gamma_4 u \frac{\partial}{\partial t} + u \frac{\partial}{\partial x}$
17	$Z^{15}(\alpha_2 \neq 0, \gamma_4 \neq 0)$	$\frac{u}{\alpha_2} - x$	$u_t + \frac{1}{\gamma_4} [1 \pm \Phi \Psi] u_x = \Phi$	$\gamma_4 u \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + \alpha_2 \frac{\partial}{\partial u}$
18	$Z^{16}(\eta_3 = \gamma_5 = 0)$	x	$u_t + [\Psi + e^{u/x}] u_x = \Phi$	$x \frac{\partial}{\partial u}$
19	$Z^{16}(\eta_3 \neq 0, \gamma_5 = 0)$	$\frac{1}{2}(\eta_3 u^2 - x^2)$	$u_t + [\Psi + e^{u/x}] u_x = \Phi$	$\eta_3 u \frac{\partial}{\partial x} + x \frac{\partial}{\partial u}$
20	$Z^{16}(\eta_3 = 0, \gamma_5 \neq 0)$	x	$u_t + [\gamma_5 \Phi^2 \pm (\gamma_5 u + x)^{\frac{1}{\gamma_5}} \Psi] u_x = \Phi$	$[\gamma_5 u + x] \frac{\partial}{\partial u}$
21	$Z^{16}(\eta_3 \neq 0, \gamma_5 \neq 0)$	B	$u_t + [\gamma_5 \Phi^2 \pm e^{x/(\eta_3 u)} + \Psi] u_x = \Phi$	$\eta_3 u \frac{\partial}{\partial x} + [\gamma_5 u + x] \frac{\partial}{\partial u}$
22	$Z^{17}(\eta_4 \neq 0)$	$\frac{x}{\eta_4} - u$	$u_t + [\eta_4 \Phi \pm u \Psi] u_x = \Phi$	$t \frac{\partial}{\partial t} + u [\eta_4 \frac{\partial}{\partial x} + \frac{\partial}{\partial u}]$
23	$Z^{17}(\eta_4 = 0)$	x	$u_t + u \Psi u_x = \Phi$	$t \frac{\partial}{\partial t} + u \frac{\partial}{\partial u}$
24	$Z^{18}(\eta_5 = \gamma_6 = 0)$	$u - \frac{x}{\eta_5}$	$u_t + \Psi u_x = u \Phi$	$u \frac{\partial}{\partial u}$
25	$Z^{18}(\eta_5 \neq 0, \gamma_6 = 0)$	$\ln u \pm x$	$u_t + \eta_5 [u \Phi + \Psi] u_x = u \Phi$	$\eta_5 u \frac{\partial}{\partial x} + u \frac{\partial}{\partial u}$
26	$Z^{18}(\eta_5 = 0, \gamma_6 \neq 0)$	x	$u_t \pm [\frac{u \Phi}{1 + \gamma_6 u \Phi} - \frac{1}{\gamma_6}] \Psi u_x = \frac{u \Phi}{1 + \gamma_6 u \Phi}$	$\gamma_6 u \frac{\partial}{\partial t} + u \frac{\partial}{\partial u}$
27	$Z^{18}(\eta_5 \neq 0, \gamma_6 \neq 0)$	$u - \frac{x}{\eta_5}$	$u_t + [\frac{\eta_5}{\gamma_6} \pm (\frac{u \Phi}{1 + \gamma_6 u \Phi} - \frac{1}{\gamma_6}) \Psi] u_x = \frac{u \Phi}{1 + \gamma_6 u \Phi}$	$\gamma_6 u \frac{\partial}{\partial t} + \eta_5 u \frac{\partial}{\partial x} + u \frac{\partial}{\partial u}$

From Proposition 3 applied to the operator Z^{14} we obtain the following additional operators for the introduced functions f and g as above

$$X^{(2)} = \gamma_3 u \frac{\partial}{\partial t} + x \frac{\partial}{\partial x}, \quad (62)$$

When $\gamma_3 \neq 0$, while for $\gamma_3 = 0$ we find $X^{(2)} = x \partial_x$ as an additional operator.

Repeating the algorithm for other Z^i s of (50), the preliminary group classification of an NIB equation of the form (4) admitting an extension \mathcal{L}_2 of the principal Lie algebra \mathcal{L}_1 is listed in Table 5. In this table we assume that

$$B = x + \gamma_3 u + \frac{\eta_1 u}{\gamma_3} \text{LambertW}(A) \quad (63)$$

where

$$A = -\frac{\gamma_3}{\eta_1 u} e^{-\gamma_3^2/\eta_1}, \quad (64)$$

and $\Phi = \Phi(\lambda)$, $\Psi = \Psi(\lambda)$ are arbitrary functions of the invariant λ .

5 Conclusion

A symmetry analysis for generalized non-homogeneous inviscid Burger's equations of class (4) led to find the structure of point infinitesimal generators as well as equivalence operators. In addition we found the modified forms of point and projective symmetries of IBE in the paper [7] among with the general form of point infinitesimals of a generalized case with respect to IBE and results of [12]. Also, equivalence classification is given of the equation admitting an extension by one of the principal Lie algebra of the equation. The paper is one of few applications of a new algebraic approach to the problem of group classification: the method of preliminary group classification. Derived results are summarized in Table 5.

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